Time Adaptive Methods for Solving Dynamic Economic Problems

Mauri Fortes[1], Wanyr R. Ferreira[2], Thales B. Maffia[1], Lamartine Sacramento Filho[1]

Resumo

Apresenta-se, neste trabalho, um método numérico adaptativo no tempo, que pode ser usado para resolver, de modo eficiente, modelos econômicos representados por equações diferenciais ordinárias. O método tem por base um esquema (resolvedor) com preditor do tipo Adams-Bashforth de segunda ordem, seguido por um preditor explícito de Euler. A partir de equações diferenciais padrões, rígidas e não-rígidas, efetuaram-se uma análise de desempenho e comparações entre o método proposto e outros procedimentos de avanço no tempo que empregam os preditores de Adams-Bashforth ou Euler e respectivos corretores. Dois modelos econômicos clássicos foram também simulados pelo modelo proposto; mostra-se a ordem de grandeza dos passos de tempo utilizados nos processos de simulação de ambos os modelos. Mostra-se, então, que o método adaptativo proposto envolvendo os esquemas de Adams-Bashforth / Euler explícito é um resolvedor excelente de equações diferenciais ordinárias e pode ser usado efetivamente na simulação de modelos econômicos dinâmicos.

Palavras-chave: Problemas Econômicos Dinâmicos- Métodos Adaptativos- EquaçõesDiferenciais - Modelagem Econômica- Modelagem Numérica

Abstract

This work presents a numerical time adaptive method that can be used efficiently to solve dynamic economic models represented by ordinary differential equations. The method consists in a second-order Adams-Bashforth predictor model followed by an explicit Euler predictor. Benchmark stiff and non-stiff ordinary differential equations were used for performance analysis and comparison of the proposed method with other time stepping procedures, that employ either Adams-Bashforth or Euler predictors and associated correctors. Two classical dynamic economic models were solved by means of the proposed model and the order of magnitude of time steps used in the simulation process is shown. The proposed Adams-Bashforth / Explicit Euler adaptive scheme is thus shown to be an excellent ordinary differential equation solver to be used in dynamic economic model simulation.

***Keywords**: Dynamic Economic Problems - Adaptive Methods - Differential Equations - Economic Modeling- Numerical Modeling*

1. UNA Business School
Rua dos Aimorés 1451 - 30140-071 - Belo Horizonte - MG - e-mail: maurif@uol.com.br
2. Mech. Eng. Dept., UFMG- Av. Antônio Carlos, 6627 - Belo Horizonte - MG

1. Introduction

Economic models are called static, when they refer to equilibrium situations, or dynamic, when they refer to situations that change with time (Weber, 1986). A model is represented by equations that correlate endogenous and exogenous variables. Rather complex models exist for analyzing business cycles (Danthine & Donaldson, 1995), derivatives in financial engineering (Banks, 1994; Wilmott et al., 1995; Wilmott, 1998) and dynamic optimization of economic problems (Judd, 1989, 1991; Benveniste & Scheinkman, 1979). The solution to the equations associated to the models can be found either analytically or numerically; analytical solutions are usually available only for simpler problems. The more complex models involve systems of partial differential equations (**PDE**) that can be solved by either finite difference (Wilmott, 1995, 1998) or weighted residual methods (Judd, 1991). Several numerical methods transform the partial differential equations into a set of ordinary differential equations (**ODE**) that must be solved efficiently. Although several techniques are available for solving ODEs, they are usually complex to handle and difficult to implement to solve PDEs. This work is concerned with the presentation of simple, effective and accurate ODE solvers. An ODE solver is said to be **adaptive** when, by means of internal error control, its time step can be automatically increased and, simultaneously, it allows the user to define the desired accuracy.

Gresho et al. (1980) proposed an implicit adaptive time integrator based on a second-order accurate **A**dams-**B**ashforth **p**redictor (**ABP**) and the trapezoid rule as the corrector. The ABP requires the evaluation of rates of change that are obtained through successive applications of the trapezoid rule. The corrector step uses again the trapezoid rule which is non-dissipative, completely stable and second-order accurate. Bixler (1989) changed Gresho et al's integrator by incorporating three modifications: 1. the one leg-twin form of the trapezoid rule replaced the trapezoid rule as the corrector, leading to a more accurate local time truncations error estimate; 2. a more stable predictor was obtained by changing the expression for obtaining rates of change and 3. the formula used for predicting time-step size was redressed to match the new corrector.

The specific objective of this work is to present a numerical time adaptive procedure to solve dynamic economic models represented by linear or non-linear ordinary differential equations and to serve as an efficient ODE solver to be coupled to

PDE solvers. The objective is centered towards further optimizing both GLS and Bixler's schemes so as to reach a more efficient algorithm. The efficiency of the proposed method was checked by means of benchmark solutions and comparisons with the previous schemes and by solving two dynamic economic models.

2. Methodology - time adaptive schemes

2.1 Bixler's implicit adaptive time integration scheme: - B scheme

Bixler (1989) modified Gresho et al's (1980) time integration scheme (GLS scheme) by incorporating modifications to enhance accuracy and stability. Given the first order differential equation (readily expandable to a system of ordinary differential equations)

$$\dot{y} = f(y, t) \tag{1}$$

Bixler's scheme (B scheme) consists basically in:

• Using the second-order-accurate Adams-Bashforth as predictor:

$$y^{p}_{n+1} = y_n + \frac{\Delta t_n}{2}\left[\left(2 + \frac{\Delta t_n}{\Delta t_{n-1}}\right)\dot{y}_n - \frac{\Delta t_n}{\Delta t_{n-1}}\dot{y}_{n-1}\right] \tag{2}$$

where the superscript p refers to predicted value and the derivatives (rates of change) at time planes n-1 and n; $\dot{y}_{n-1}$ and $\dot{y}_n$ are respectively approximated by:

$$\dot{y}_{n-1} = \frac{\Delta t_{n-2}}{\Delta t_{n-1} + \Delta t_{n-2}}\left(\frac{y_n - y_{n-1}}{\Delta t_{n-1}}\right) + \frac{\Delta t_{n-1}}{\Delta t_{n-1} + \Delta t_{n-2}}\left(\frac{y_{n-1} - y_{n-2}}{\Delta t_{n-2}}\right) \tag{3}$$

and the trapezoid rule:

$$\dot{y}_n = \frac{2}{\Delta t_{n-1}}\left(y_n - y_{n-1}\right) - \dot{y}_{n-1} \tag{4}$$

• Using the one-leg twin of the trapezoid rule as corrector:

$$\frac{y_{n+1} - y_n}{\Delta t_n} = f\left(\frac{y_{n+1} + y_n}{2}, \frac{t_{n+1} + t_n}{2}\right) \quad (5)$$

• Predicting the time step size by means of the expression:

$$\Delta t_{n+1} = \Delta t_n \left(\frac{\varepsilon}{|d_{n+1}|}\right)^{\frac{1}{3}} \quad (6)$$

where ε is a target local time-truncation error and d_{n+1} is the distance between the corrected and exact solution and is defined by

$$d_{n+1} = \frac{0.25}{2.25 + 3\Delta t_{n-1} / \Delta t_n}\left(y_{n+1} - y^p_{n+1}\right) \quad (7)$$

Taking the superscript E to refer to the exact solution, the local time truncation error estimate relative to equation (2) can be obtained by means of Taylor series expansion as (Gresho et al., 1980)

$$y_{n+1} - y^E_{n+1} = \frac{1}{12}\Delta t_n^3 \ddot{f}(y,t) + O(\Delta t^4) \quad (8)$$

Similarly, the local truncation error for equation (5) is given by (Bixler, 1989):

$$y_{n+1} - y^E_{n+1} = \frac{0.25}{12}\Delta t_n^3 \ddot{f}(y,t) + O(\Delta t^4) \quad (9)$$

Combination of the above two equations (8-9) allows obtaining the time step size by equation (6). It should be noted that the B scheme requires solutions at three preceding time steps. Adaptive time stepping can start at the fourth step. Furthermore, a norm such as the root means-square should replace the absolute norm in equations (6-7) when a system of equations is to be solved.

2.2 GLS - Euler integration formulas: GLS-E scheme

GLS (Gresho et al, 1980) suggested a scheme using the explicit forward Euler (FE) as the predictor and, as the corrector, the implicit backward Euler (BE), respectively given by

$$y^{P}_{n+1} = y_n + \Delta t_n f(y_n) = y_n + \Delta t_n \dot{y}_n \text{ and } y_{n+1} = y_n + \Delta t_n f(y_{n+1}) \qquad \textbf{(10)}$$

Furthermore, $\dot{y}_n$ in equation (10) was suggested to be obtained from the inverted form of the BE scheme:

$$\dot{y}_{n+1} = \frac{y_{n+1} - y_n}{\Delta t} \qquad \textbf{(11)}$$

GLS showed that the local truncation errors for the two schemes are, respectively:

$$y_{n+1} - y^{E}_{n+1} = -\frac{\Delta t_n^2 \dot{f}_n(y,t)}{2} + O(\Delta t^3) \text{ and}$$

$$y_{n+1} - y^{E}_{n+1} = d_{n+1} = \frac{\Delta t_n^2 \dot{f}_n(y,t)}{2} + O(\Delta t^3) \qquad \textbf{(12)}$$

The last two equations lead directly to the expression for the next step size:

$$\Delta t_{n+1} = \Delta t_n \left(\frac{\varepsilon}{|d_{n+1}|} \right)^{\frac{1}{2}} \text{ where } d_{n+1} = \frac{1}{2}\left(y_{n+1} - y^{P}_{n+1}\right) + O(\Delta t^3) \qquad \textbf{(13)}$$

For both the second and first-order schemes, GLS recommend the one-step Newton-Raphson method to solve the implicit correctors. Following GLS, the GLS-Euler scheme, because of its dissipative nature, should be used only as a means for obtaining steady-state solutions.

2.3 Candidate schemes

Several time adaptive integration techniques were investigated in a search for adequate methods for solving dynamic economic problems. The following methods were tested:

1. *Second order scheme:* ***AB-E*** *scheme.* Instead of using the trapezoid rule (GLS) or the one-leg twin of the trapezoid rule (Bixler), the explicit Euler equation was checked as a corrector for Adams-Bashforth method. Thus, the method consisted of Adams-Bashforth predictor associated to the explicit Euler corrector (the time step size was obtained from equation (6)):

$$y_{n+1} = y_n + \Delta t_n f(y^P_{n+1}) \quad \textbf{(14)}$$

2. *Second order scheme: - modified-GLS (****MGLS****).* Here, instead of using equation(5), the explicit form of the trapezoid rule was used as corrector. No use was made of the Newton-Raphson method. The rest of the procedures follows Bixler's scheme.

3. *First order schemes:* Two schemes were tested: a) ***GLS-E*** and b) ***EE***: - Fully explicit Euler schemes, which consists in using the FE scheme as predictor, equation(10), and the explicit form of the Euler given by equation (14).

4. *Mixed first and second-order scheme:- Modified Euler - Trapezoid rule (****MET****).* Another possibility is to employ a modified Euler predictor-corrector method, MET, which consists in using the FE scheme as predictor and the trapezoid rule as the corrector, in the form:

$$y_{n+1} = y_n + \frac{1}{2}\Delta t_n \left[f_n + f^P_{n+1} \right] \quad \textbf{(15)}$$

The trapezoid rule is a second order scheme and the FE is first order. However, the resulting predictor-corrector method is second-order (Hoffman, 1992), when applied to ordinary linear differential equations. Because the predictor and corrector do not possess the same order, an expression for predicting the time step is not readily available.

By assuming that Δt_n and $\Delta y^* = |y_{n+1} - y^P_{n+1}|$ are the same for both the GLS-E and Bixler's methods, it can be concluded that as long as $\Delta y^* > 2.14 \times 10^{-5}$ the step sizes predicted by Bixler's method are larger than the ones predicted by the GLS-E method. Consequently, based on this argument, and the order of the candidate schemes, it can be concluded that any of the time step predicting equations could be used with any of the candidate schemes. However, in the following test problem, the time step sizes for the B, AB-E, MGLS and MET schemes were evaluated by means of equation (6) while the step sizes for the first order schemes, GLS-E and E-E, were evaluated by means of equation (13).

3. Results

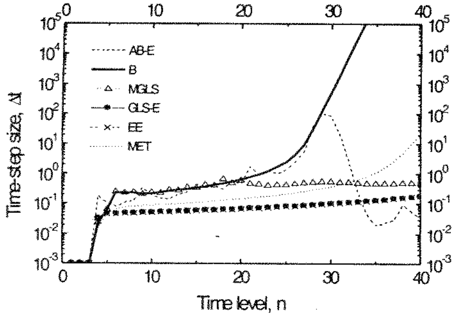
3.1 Stiff and non-stiff problems

Here Bixler's (1989) test problem is used: $\dot{y} = -y^k$. The relevant behavior of the time integrators is illustrated by considering values of $k = 1$ and 2, which provide stiff and non-stiff responses, respectively. In both cases, the initial condition was taken to be $y = 1$ at $t = 0$ and the initial time step was taken to be $\Delta t = 10^{-3}$. Some of the proposed schemes could, in principle, be discarded based on simple error analysis and final conclusions by both Bixler (1989) and Gresho et al. (1980). However, solvers for linear and non-linear problems can behave differently according to boundary values and to the associated parameters (Hoffman, 1992) and, thus, the results to be presented may be of value for eventual comparison purposes.

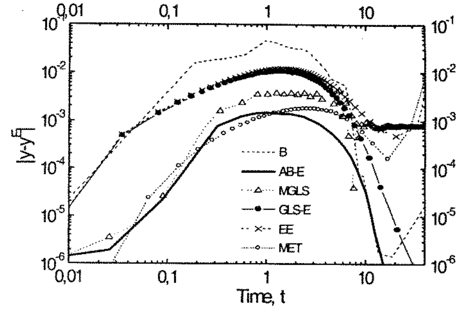
Figure 1 shows numerical results for the equation $\dot{y} = -y$. It can be observed that:

- At $t = 20$, the exact solution is approximately 2.06×10^{-9} which can be considered, for most purposes, close enough to the asymptotic solution ($y = 0$).

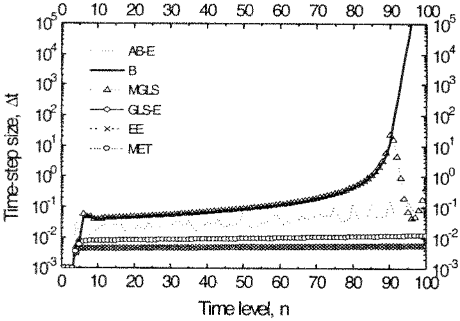
- The best overall performance was obtained by means of Bixler's (**B**) scheme. However, as mentioned before, **B** scheme is implicit.



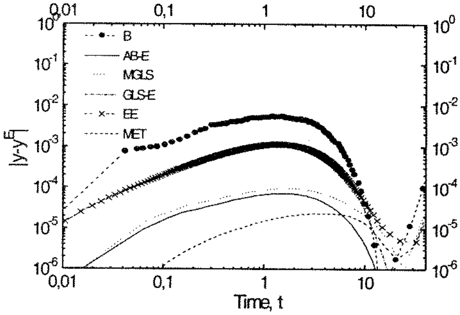
(a) Time-step sizes for $\varepsilon = 10^{-3}$



(b) Errors for $\varepsilon = 10^{-3}$



(c) Tim· step sizes for $\varepsilon = 10^{-5}$



(d) Errors for $\varepsilon = 10^{-5}$

Figure 1. Time-step sizes and errors for different schemes and error levels, when solving $\dot{y} = -y$.

- The MGLS is more effective when higher accuracy is desired, as compared to the AB-E scheme; however, if the expected error is $\varepsilon = 10^{-3}$, the AB-E scheme is much more efficient. The GLS-E and EE schemes behaved almost identically in the time range investigated (the GLS-E scheme is more accurate).

- The MET scheme showed a performance better than the first-order and inferior to the AB-E schemes; it should be mentioned that the MET showed oscillatory results for longer simulation times.

Figure 2 shows numerical results for the equation $\dot{y} = -y^2$. It can be observed that:

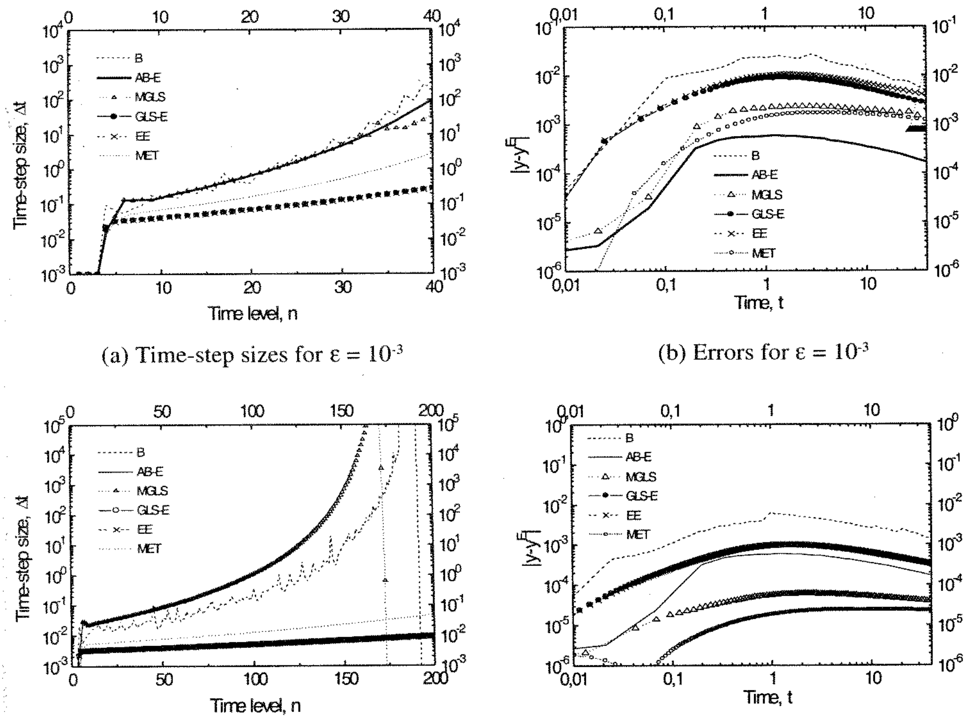
- When high accuracy is desired, in the present case $\varepsilon = 10^{-5}$, the first order schemes (GLS-E and EE) and the MET scheme were rather inefficient as compared to the B, AB-E and MGLS schemes.

- Again, the best overall performance was obtained by means of Bixler's scheme; nevertheless, the AB-E and MGLS schemes presented excellent performance.

- The MET scheme is rather slow at higher specified ε; however it remains an alternative when compared to the B scheme, due to its simplicity.

Thus, based on the previous stiff and non-stiff examples, it can be said that the Adams-Bashforth method followed by either the explicit forward Euler or the explicit trapezoid rule (AB-E and MGLS) constitute acceptable alternatives to the implicit Bixler and GLS schemes. Although less efficient, the EE and MET schemes should be considered as eventual explicit substitutes to the just cited schemes. The implicit GLS scheme, since it is implicit, can be replaced by the explicit counterpart, the EE scheme.

Based on the just exposed reasoning and data and on the fact that it is fully explicit ($\theta = 0$), the AB-E scheme will be applied to solve two economic models and the results will be compared with analytical solution.



(a) Time-step sizes for $\varepsilon = 10^{-3}$

(b) Errors for $\varepsilon = 10^{-3}$

(c) Time-step sizes for $\varepsilon = 10^{-5}$

(d) Errors for $\varepsilon = 10^{-5}$

Figure 2. Time-step sizes and errors for different schemes and error levels, when solving $\dot{y} = -y^2$.

3.2 Application of the AB-E scheme to simple macroeconomic problems

Domar's macro model - Domar's model expresses the relationship among income, saving and investment and is given by

$$S(t) = \alpha y(t) \qquad \text{and} \qquad I(t) = \beta \frac{dy}{dt} \tag{16}$$

where S is savings, I is investment, y is income and all of these endogenous variables are functions of time, t. The parameters $\alpha > 0$ and $\beta > 0$. The initial condition is $y(0) = y_0$. Since $S(t) = I(t)$ equations (16) lead to

$$\frac{dy}{dt} - \frac{\alpha}{\beta} y = 0 \tag{17}$$

that has the analytical solution $y = y_0 e^{(\alpha/\beta)t}$.

Figure 3 shows the solutions obtained by means of the AB-E method and the analytical solution. The AB-E method leads to a quasi-exact solution (maximum $\varepsilon = 0.5\%$), and allows the time-step size to increase so that the time final value is reached in 440 steps whereas Euler's scheme behaves much more inefficiently, requiring 1.8×10^5 steps.

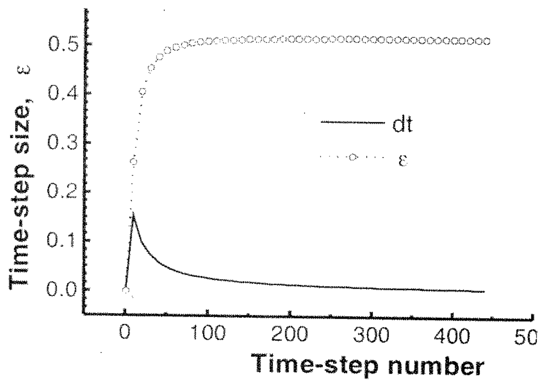


Figure 3. Time-step sizes and percent relative error using AB-E scheme for the Domar's macromodel.

Domar's debt model - Domar's debt model expresses the relationship between the national income and the national debt and is represented by:

$$\frac{dD}{dt} = \alpha y(t) \qquad \text{and} \qquad \frac{dy}{dt} = \beta \tag{18}$$

where D is the national debt, y is the national income and these endogenous variables are functions of time, t. The parameters $\alpha > 0$ and $\beta > 0$ .The initial conditions are $y(0)=y_0$ and $D(0)=D_0$. The analytical solution to this problem is given by

$$D(t) = \frac{1}{2}\alpha\beta t^2 + \alpha y_0 t + D_0 \qquad \text{and} \qquad y(t) = \beta t + y_0 \tag{19}$$

Figure 4 shows the solution obtained by means of AB-E scheme. The relative difference between analytical and numerical solutions did not exceed $1.5\times10^{-4}\%$. We can note that the AB-E allows the time-step size to increase in such a way that at 7 time-steps it increases by a factor of 5×10^5, starting from 1×10^{-3} and reaching the value of 500.

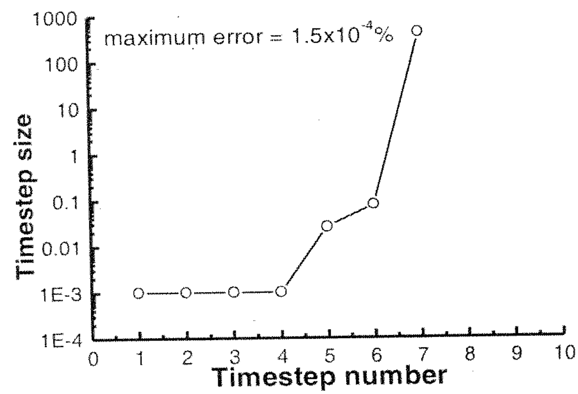


Figure 4. Time-step sizes and percent relative error using AB-E scheme for the Domar's debt model.

4. conclusions

This work presents a numerical time adaptive method based on Adams-Bashforth(Predictor) and Euler (Corrector) to solve dynamic economic models represented by ordinary differential equations. A performance analysis of this method was made and compared with other time stepping procedures. Two classical dynamic economic models are solved utilizing the proposed method and the results are compared with analytical solutions.

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