ARQUIVO 4

Comparison between the Diffusional Finite Difference and the Radial Basis Function Methods when Applied to Financial Engineering

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Abstract

Mathematical modeling of the famous Black and Scholes equation has become essential tools in the financial industry. However, the numerical solution of the various forms of Black and Scholes equation has been a long standing problem in spite of the many available numerical methods. In this paper, we present a comparison between a diffusional finite-difference method and the meshless method based on thin-plate and cubic radial basis functions (RBF). The analysis included the effects of the time integration method, mesh size, time step, maximum simulated stock value, volatility and interest rate on the stability and accuracy of solutions of call options. The results show that the finite difference solutions are very stable but much less accurate than the ones obtained by means of the RBF method. The RBF methods, although subject to divergence, are very accurate and simple to implement.

Key-words. Black-Scholes, financial engineering, radial basis functions, diffusional method, finite difference, numerical methods.

Comparação entre os Métodos Difusional de Diferenças Finitas e Funções de Base Radial quando aplicados à Engenharia Financeira

Resumo

A modelagem matemática da famosa equação de Black-Scholes tornou-se um instrumento indispensável para a indústria financeira. Entretanto, a solução numérica das várias formas da equação de Black e Scholes se transformou num problema permanente apesar dos muitos métodos numéricos disponíveis. Neste artigo, apresenta-se uma comparação entre um método difusional acoplado a diferenças finitas e o método que não requer malhas, baseado nas funções de base radial (RBF),Thin-Plate Spline e cúbica. A análise incluiu os efeitos do método de integração temporal, tamanho de malha, valor máximo simulado da ação, volatilidade e taxa de juros sobre a acurácia e estabilidade das soluções de opções de compra. Os resultados mostram que as soluções por diferenças finitas são muito estáveis, mas muito menos acuradas que as obtidas pelo método RBF. Este método, embora sujeito a divergências, é muito acurado e fácil de ser implementado.

Palavras-chave. Black-Scholes, engenharia financeira, funções de base radial, método difusional, diferenças finitas, métodos numéricos.

Introduction

The pioneering work of Black and Scholes (1973), BS, started the serious study of the theory of option pricing. All further advances in this field have been extensions and refinements of the original idea expressed in that paper (MEYER & JOHNSON, 2002). The Black-Scholes option-pricing model is used to determine the expected value of an option. It provides insight into the valuation of debt relative to equity (HULL, 1989; SIEGEL et al., 1992). Mathematical modeling and simulation based on the famous Black and Scholes equation (or BS equation) have become indispensable tools in the financial industry. It is well known that banks, brokerage houses and investors and their consulting services spend countless hours every day and night to simulate and predict the movement of prices for financial assets like stocks, options and bonds. Much of the mathematics employed in this field, particularly in academic research, is highly sophisticated and spans the fields of analysis, probability, statistics, differential equations and, last but not least, numerical analysis (COX & RUBINSTEIN, 1985; MEYER, 1998; LEENTVAAR, 2003; MUCHMORE et al., 2005; CRETIEN, 2006).

This work aims to discuss highly accurate numerical methods to be used to solve the BS equation. Although only European call options are analyzed, the main reason for the work is to propose general numerical models for many different types of options. In particular, American options are not solvable in an analytic sense (MEYER & JOHNSON, 2002). If a numerical method works for European style option, then this should the basis to get the solution for an American option. Another issue is "implied volatility". Volatility is a quantitative expression for the randomness in the market. From newspapers or stock exchanges, the volatility of asset prices in the future is not known, so it has to be estimated. If the model leads to accurate values for the option price, it is possible to calculate the volatility, which is the only unknown parameter in the Black-Scholes equation.

Most of the financial engineering (area that deals with creative application of financial theory and instruments) cases require numerical methods to compute an approximate solution (BARUCCI et al., 1996; GARCIA-OLIVARES, 2003; ZHANG, 2005). It is very difficult to generate stable and accurate solutions to Black-Scholes equations due to the discontinuity of the pay-off function around the exercise price (KOC et al., 2003; CONT & VOLTCHKOVA, 2005). Thus, many numerical methods have been introduced to model accurately the interaction between advective and diffusive behaviors present in the different modified forms of the BS equation. Finite difference and finite element solutions of the advection-diffusion equation, a general statement of the BS equation, present numerical problems of oscillations and damping (BOZTOSUN & CHARAFI, 2002; HOFFMAN, 1992; MURPHY & PRENTER, 1985; WILMOTT, 1998; WILMOTT et al., 1995; LEE et al., 1987; ZIENKIEWICZ & TAYLOR, 1991). However, these traditional methods have numerical problems of oscillations and damping and require the mesh generation procedures (KOC et al., 2003).

The mesh generation problem over irregularly shaped domains is often in excess of 70% of the total computational cost (BROWN et al., 2005). Thus, meshless methods such as those using Radial Basis Functions (RBFs) have, therefore and recently, attracted much attention of the engineering community and RBFs have become an important tool in scientific computing (KARA-

GEORGHIS, 2006). Radial basis function (RBF) methods have shown the potential to be a universal grid-free method for the numerical solution of partial differential equations (SARRA, 2004). In this work, we consider the use of the thin plate spline radial basis function (TPS RBF), the cubic radial basis function (Cubic RBF) and a literature claimed accurate finite difference methodology. In their work with radial basis functions, RBF, Boztosun and Charafi (2002) reason on the fact that previous analyses have shown that the multiquadrics (MQ) and TPS give the most accurate results for scattered data approximations. However, these authors point that the accuracy of the MQ method depends on a shape parameter and as yet there is no mathematical theory about how to choose its optimal value. Hence, most applications of the MQ use experimental tuning parameters or expensive optimization techniques to evaluate the optimum shape parameter, while the TPS method gives good agreement without requiring such additional parameters and based on sound mathematical theory. On the other hand, Boztosun and Charafi (2002) have not analyzed the use of both Cubic RBF and TPS RBF to solve the BS equation.

The diffusional method was proposed by Fortes (1997) and applied to the solutions of several benchmark problems (FORTES & FERREIRA, 1998 and 1999) and is based on transforming original hyperbolic parabolic partial differential equation into a parabolic partial differential equation.

A recent work, Marques and Fortes (2007) analyzed several aspects of the solution of the diffusional and non-diffusional forms of the BS equation by means of TPS and Cubic RBF. Their main conclusions were:

- In the context of the RBF method, the diffusional method and the classical form of the BS equation lead to the same results and can be used interchangeably.
- Both TPS and Cubic radial functions furnish accurate results to the BS equation. There is no definitive evidence of sensitive differences between both methods.

- *Excellent results can be obtained through:*
    - Using an integration method with ? ? 0,5 (in other words, use an implicit proce dure including and between the Crank-Nicolson and Euler fully implicit scheme);
    - Using a high number of time steps in the case of the TPS RBF and the lowest convergent number for Cubic RBF;
    - Using the minimum convergent value for the stock mesh size for both RBF. In other words, one should simulate the call option situation, decrease the mesh size and use the smallest mesh size that allows convergence.
    - Using a maximum value for the stock price approximately equal to two times the exercise price

To establish the full performance of the RBF method, more stringent numerical analyses are required then the ones effectuated by Marques and Fortes (2007). Similarly, up to present, no assessment has been made on the accuracy of the diffusional finite difference method under fluctuations of interest rate and volatility. More specifically this paper is concerned with:

1. Solving and comparing the diffusional finite difference and the RBF methods as to their ability to solve the BS equation, with respect to accuracy and solution stability.

2. Effectuating parametric sensitivity analyses by investigating the effect of variations of interest rate and volatility on the accuracy of the numerical methods.

Methodology

The diffusional method for solving convection-diffusion equation
The transient one-dimensional convection-diffusion equation can be written in the non-conservative form as

$$\frac{\partial \hat{V}}{\partial t}+u\frac{\partial \hat{V}}{\partial S}-\frac{\partial}{\partial S}\Gamma\frac{\partial \hat{V}}{\partial S}+Q=0 \qquad (1)$$

where ^v is the dependent variable, and the parameters Q, u and T may depend on ^v, t and S. Fortes (1997) and Fortes & Ferreira (1999) showed that the above equation can be put into the following diffusional form

$$\frac{\partial \hat{V}}{\partial t}-e^{\int_0^s \frac{u}{\Gamma}dS}\frac{\partial}{\partial S}\left(\Gamma e^{-\int_0^s \frac{u}{\Gamma}dS}\frac{\partial \hat{V}}{\partial S}\right)+Q=0 \qquad (2)$$

If one assumes u/T to be constant or an average within the integration range, which is from 0 to L, then the above equation can be written in terms of the global Peclet number, Pe=u/T, as,

$$e^{\frac{2Pe\,S}{L}}\frac{\partial \hat{V}}{\partial t}-\frac{\partial}{\partial S}\left(\Gamma e^{\frac{2PeS}{L}}\frac{\partial \hat{V}}{\partial S}\right)+e^{\frac{2Pe\,S_L}{L}}Q=0 \qquad (3)$$

The above equation (3) is in an excellent form, suited to be solved by any numerical technique, and more particularly, by the finite difference method.

The finite difference diffusional PMGV (Prevailing Main Grid Value) scheme

Consider the discretization diagram shown in Fig. 1.

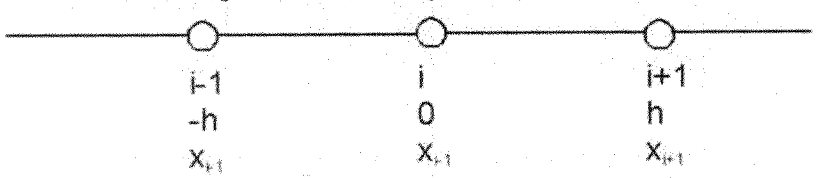

Figure 1. The discretization nodes and respective spatial coordinates.

If a central difference approximation is used and uniform grid spacing is assumed, then the diffusive term of Eq. (3) can be written as

$$\frac{\partial}{\partial S}\left(\Gamma e^{-\frac{2PeS}{h}}\frac{\partial V}{\partial S}\right)_i \approx \frac{1}{h}\left[\left(\Gamma e^{-\frac{2PeS}{h}}\frac{\partial V}{\partial S}\right)_{i+1/2} - \left(\Gamma e^{-\frac{2PeS}{h}}\frac{\partial V}{\partial S}\right)_{i-1/2}\right] \quad (4)$$

where h = ▲S and the subscript i + ½ indicates an evaluation at the point midway x_i and x_{i+1} and the subscript i - ½ is similarly defined. Thus, we can write

$$\frac{\partial}{\partial S}\left(\Gamma e^{-\frac{2PeS}{h}}\frac{\partial V}{\partial S}\right) \approx \frac{1}{h}\left[\left(\Gamma e^{-Pe}\frac{\partial V}{\partial S}\right)_{i+1/2} - \left(\Gamma e^{Pe}\frac{\partial V}{\partial S}\right)_{i-1/2}\right] \quad (5)$$

Again, by assuming constant local properties, replacing the appropriate derivatives by central difference approximations, and rearranging, we obtain

$$\frac{\partial}{\partial S}\left(\Gamma e^{-\frac{2PeS}{h}}\frac{\partial \hat{V}}{\partial S}\right) \approx \frac{\Gamma}{h^2}\left[e^{-Pe}V_{i+1} - \left(e^{-Pe} + e^{Pe}\right)V_i + e^{Pe}V_{i-1}\right] \quad (6)$$

The source term can be approximated by applying the log-mean difference concept to the exponential term multiplying Q. Thus

$$e^{-\frac{2Pe\,S}{h}}Q \approx \frac{e^{Pe} - e^{-Pe}}{2Pe}Q \quad (7)$$

The transient term is approximated by the classical finite difference technique with the log-mean difference concept being applied to the exponential term

$$e^{-\frac{2PeS}{h}}\frac{\partial V}{\partial t} \approx \left(\frac{e^{Pe} - e^{-Pe}}{2Pe}\right)\left(\frac{V_i^{n+1} - V_i^n}{\Delta t}\right) \quad (8)$$

where n stands for time plane. By making use of the mean value theorem, Eqs. (6, 7 and 8) can be substituted into Eq. (3), resulting in a set of algebraic equations whose ith-equation is given by

$$\left(\frac{V_i^{n+1}-V_i^n}{\Delta t}\right)+\theta\left\{-\frac{2Pe\Gamma}{h^2}\left[\left(\frac{e^{Pe}}{e^{Pe}-e^{-Pe}}\right)V_{i-1}^{n+1}-(\coth Pe)V_i^{n+1}+\left(\frac{e^{-Pe}}{e^{Pe}-e^{-Pe}}\right)V_{i+1}^{n+1}\right]+Q^{n+1}\right\}$$
$$+(1-\theta)\left\{-\frac{2Pe\Gamma}{h^2}\left[\left(\frac{e^{Pe}}{e^{Pe}-e^{-Pe}}\right)V_{i-1}^{n}-(\coth Pe)V_i^{n}+\left(\frac{e^{-Pe}}{e^{Pe}-e^{-Pe}}\right)V_{i+1}^{n}\right]+Q^{n}\right\}=0$$
(9)

where 0 <= 0 <=1.
Simple algebraic manipulations and the use of the& parameter lead to the coefficients in Table 1.

After spatial and one-step time discretization of the above transient convection-diffusion equation, one arrives at the classical approximate equation,

$$\frac{\mathbf{C}\left(V^{n+1}-V^{n}\right)}{\Delta t}+\mathbf{K}\left[(1-\theta)V^{n}+\theta V^{n+1}\right]+\mathbf{D}^{n}=0$$
(10)

where 0 <= 0 <=1; C, K and D are, respectively, the capacitance, the conductivity or stiffness matrices and the force vector, and n is the time level. If the problem at hand is one-dimensional, if use is made of the terminology such that as, bs and cs refer to the steady state terms, while at, bt and ct refer to the transient terms, then equation (10) can be rewritten in the form

$$(a_t+a_s\theta)V_{i-1}^{n+1}+(b_t+b_s\theta)V_i^{n+1}+(c_t+c_s\theta)V_{i+1}^{n+1}=$$
$$-d_i+[a_t-a_s(1-\theta)]V_{i-1}^{n}+[b_t-b_s(1-\theta)]V_i^{n}+[c_t-c_s(1-\theta)]V_{i+1}^{n}$$
(11)

where the terms defining the coefficients are given in Table 1. By means of an integration of the transient term assuming that only the main grid point value matters (the i_th nodal value), as is done in the Finite Volume Method (PATANKAR, 1980), then one arrives at only a b_t coefficient, that is

$$b_t^*=a_t+b_t+c_t \qquad a_t^*=0 \quad \text{and} \quad c_t^*=0$$
(12)

This new formulation has been called the **Prevailing Main Grid Value (PMGV)** scheme and can be thus written as:

$$a_s\theta V_{i-1}^{n+1}+(b_t+b_s\theta)V_i^{n+1}+c_s\theta V_{i+1}^{n+1}=$$
$$-d_i-a_s(1-\theta)V_{i-1}^{n}+(b_t-b_s(1-\theta))V_i^{n}-c_s(1-\theta)V_{i+1}^{n}$$
(13)

The PMGV scheme shows the property that if an explicit scheme is used, the capacitance matrix becomes diagonalized and no solver is required for obtaining the new V values.

Table 1 - Coefficients for the transient one-dimensional diffusional scheme

$\lvert\alpha\rvert = \alpha_{opt} = \coth\lvert Pe\rvert - \frac{1}{\lvert Pe\rvert}$	$d_i = \frac{Qh^2}{\Gamma}$	$b_t^* = a_t + b_t + c_t = \frac{2Pe}{C}$
$a_s = -Pe(\alpha+1)-1$	$b_s = 2 + 2Pe\alpha$	$c_s = -Pe(\alpha-1)-1$
$a_t = \frac{\alpha}{C}[Pe(\alpha+1)+1]$	$b_t = -\frac{2}{C}[Pe(\alpha^2-1)+\alpha]$	$c_t = \frac{\alpha}{C}[Pe(\alpha-1)+1]$

PMGV solution methodology for Black-Scholes Equation for Call Options

The classical form of the basic Black-Scholes equation, BS, is (WILMOTT, 1998):

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \quad (14)$$

Where V,t,&, S and r stand, respectively, for option value (price), time, volatility, asset (underlying security) price (a stochastic variable) and interest rate

In this work, the numerical calculations involved solving the BS equation with a call option with the following payoff function, that is, the value of the call option at expiry (t =T), in a neutral-risk world:

Payoff (S, T) = max (S-E, 0) (15)

where E is the option exercise or strike value, that is, its value at t = T. The respective boundary conditions are:

$$V(0, \tau) = 0 \quad ; \quad V(\infty,\tau) = S - Ee^{-r(T-\tau)} \quad (16)$$

One should note that this is not an initial value problem, since the payoff function is given at t = T. In order to make it an initial boundary value problem let us make t = T - t, so that the above equation becomes

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0 \quad (17)$$

In order to put this last equation in the form expressed by equation (1), use is made of the identity

$$\frac{\partial}{\partial S}\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S}\right)=\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}+\sigma^2 S\frac{\partial V}{\partial S} \tag{18}$$

in equation (17), so that one obtains

$$\frac{\partial V}{\partial \tau}-\frac{\partial}{\partial S}\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S}\right)+\left(\sigma^2 - r\right)S\frac{\partial V}{\partial S}+rV=0 \tag{19}$$

where one can easily identify

$$u=\left(\sigma^2 - r\right)S; \qquad \Gamma=\frac{1}{2}\sigma^2 S^2; \qquad Q = rV \qquad Pe=\frac{\left(\sigma^2 - r\right)\Delta S}{\sigma^2 S} \tag{20}$$

Where ▲S can, now, be identified as a characteristic length (spot market price and, in this case, the mesh size). The initial and boundary conditions are:

$$V(S, 0) = \text{Payoff}(S,0) = \text{Max}\ (S-E, 0); \quad V(0, \tau) = 0\ ;\ V(\infty,\tau) = S - Ee^{-r\tau} \tag{21}$$

Sensitivity and parametric analysis

Equation (20) depicts clearly the importance of sensitivity parametric analysis. In this work, sensitivity analysis is defined as an analysis around optimized (minimized or maximized) function values, in this case, option values. On the other hand, parametric analysis is defined as the analysis made by changing the important parameters; in this work, the interest rate and volatility.

In countries like Brazil, large risk-free interest rates may occur. Under international economic crash imminent conditions, large volatility variations are probable. U, and Pe can be highly affected by these changes and thus, as well, the solutions might be very different.

For the above mentioned reasons, simulations were undertaken for different risk-free interest rates and for different volatilities.

Radial basis functions applied to the original Black-Scholes equation

The idea behind the RBF method is to use linear translate combinations of a basis function $\phi(\mathbf{r})$ of one variable, expanded about given scattered 'data centers' to approximate an unknown function $S_i \in \Re^d, i=1,\ldots,N$ by

$$V(S,t)=\sum_{j=1}^{N}\lambda_j(t)\phi(r_j)=\sum_{j=1}^{N}\lambda_j\phi(\|S-S_j\|) \quad (22)$$

where rj = $\|S-S_j\|$ is the Euclidean norm and are the coefficients to be determined. Usual radial basis functions are defined by (KOC et al., 2003):

$$\text{Thin-Plate Spline, TPS}: \phi(r_j)=r_j^4\log(r_j) \quad (23)$$

$$\text{Multiquadrics, MQ}: \phi(r_j)=\sqrt{c^2+r_j^2} \quad (24)$$

$$\text{Cubic}: \phi(r_j)=r_j^3 \quad (25)$$

$$\text{Gaussian}: \phi(r_j)=e^{-c^2r_j^2} \quad (26)$$

In this work, only cubic and TPS RBF will be used, due to their simplicity and proven accuracy for other types of problems and the difficulty associated to choosing good values for the shape parameter c, which depends on the problem type (BOZTOSUN and CHARAFI, 2002).

The original Black-Scholes equation shown above in equation (19) can be discretized using the ϕ -weighted method:

$$\frac{\partial V(S,t)}{\partial t}=f(S,t)\approx(1-\theta)\cdot f(S_t,t)+\theta\cdot f(S_{t+\Delta t},t+\Delta t) \qquad \text{for } 0?\ \theta\ ?1 \quad (27)$$

So, equation (19) becomes:

$$\begin{aligned}&V(S,t)-V(S,t+\Delta t)+\Delta t(1-\theta)\cdot\left[\frac{1}{2}\sigma^2S^2\nabla^2V(S,t)+rS\nabla V(S,t)-rV(S,t)\right]+\\&+\Delta t\theta\cdot\left[\frac{1}{2}\sigma^2S^2\nabla^2V(S,t+\Delta t)+rS\nabla V(S,t+\Delta t)-rV(S,t+\Delta t)\right]=0\end{aligned} \quad (28)$$

Or

$$\begin{aligned}&V(S,t^n)\cdot\left[1+\Delta t(1-\theta)\cdot\left(\frac{1}{2}\sigma^2S^2\nabla^2+rS\nabla-r\right)\right]+\\&+V(S,t^n+\Delta t)\cdot\left[-1+\Delta t\theta\cdot\left(\frac{1}{2}\sigma^2S^2\nabla^2+rS\nabla-r\right)\right]=0\end{aligned} \quad (29)$$

Defining $V(\mathbf{S},t^n)=V^n$ and $V(\mathbf{S},t^n+\Delta t)=V^{n+1}$, then the previous equation can be written in the form:

$$\left[1-\alpha\cdot\left(\frac{1}{2}\sigma^2 S^2\nabla^2+rS\nabla-r\right)\right]\cdot V^{n+1}=\left[1+\beta\cdot\left(\frac{1}{2}\sigma^2 S^2\nabla^2+rS\nabla-r\right)\right]\cdot V^{n} \tag{30}$$

where $\alpha = \theta\Delta t$, $\beta = (1-\theta)\Delta t$, $\nabla=\frac{\partial}{\partial S}$ and $\nabla^2=\frac{\partial^2}{\partial S^2}$. And, now, by defining two new operators, H_+ and H_-:

$$H_+=1-\alpha\cdot\left(\frac{1}{2}\sigma^2 S^2\nabla^2+rS\nabla-r\right),\quad H_-=1+\beta\cdot\left(\frac{1}{2}\sigma^2 S^2\nabla^2+rS\nabla-r\right) \tag{31}$$

equation (14) becomes:

$$\sum_{j=1}^{N}\lambda_j^{n+1}H_+\phi(S_{ij})=\sum_{j=1}^{N}\lambda_j^{n}H_-\phi(S_{ij}) \qquad \text{for } i = 1\ldots N \tag{32}$$

Equation (32) generates a system of linear equations, which can be solved to obtain the unknowns, λ_j^{n+1}, from the known values of λ_j^n at a previous time step. Then they are transformed to the V(S, t) by equation (22).

Analytical solution

The analytical solution for the present value of a call option is given by:

$$V(S,T-t)=S\cdot N(d_1)-E\cdot N(d_2)\cdot e^{r(T-t)} \tag{33}$$

where

$$d_1(S,T-t)=\frac{\ln\left(\frac{S}{E}\right)+\left(r+\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}} \tag{34}$$

$$d_2(S,T-t)=d_1(S,T-t)-\sigma\cdot\sqrt{T-t} \tag{35}$$

and N is the cumulative normal probability density function,

$$N(x)=(2\pi)^{-\frac{1}{2}}\int_0^x e^{-\frac{1}{2}u^2}du+\frac{1}{2}. \tag{36}$$

The instantaneous Payoff function, M, is given by

$$M(S,T-t))=\begin{cases} S-Ee^{-r(T-t)}, \text{if } S-Ee^{-r(T-t)}\geq 0 \\ 0, \text{otherwise}\end{cases} \tag{37}$$

Results and Discussion

The results to be shown were obtained via Mathcad, a symbolic mathematical programming language and solver. Option value or price or premium is defined as the call option value that an option buyer pays the seller. The reference data used in the simulation studies were:

- Exercise value = E = 50
- Volatility =   = 20%
- Riskless interest rate = r = 5%
- Expiry time = T = 1
- Present exact analytical call option value = 5.225

The total number of stock price meshes is N and the mesh size, S, is defined by S = S/N. The total number of time steps is Nt, while the time step, t, is defined by t = T/Nt.
In this work, numerical option value relative errors refer to option prediction values at the strike (exercise) price value (S = E = 50) and are defined as:

$$\varepsilon(\%) = \frac{\text{Numerical option value - Analytical solution value}}{\text{Analytical solution value}} \times 100\% \qquad (38)$$

One of the boundary conditions, typical in BS problems, requires specifying V(S, t) at S =?; practical numerical solutions require that this value should be reduced and, the larger the allowable reduction, the better the efficiency of the numerical solution, due to decreased equation matrix size; thus, the practical maximum simulated value for S was called Smax.

The accuracy of finite difference solutions of BS equations can be heavily improved if the diffusional method substitutes the classical approach (FORTES et al, 2005). However, in a previous work (MARQUES and FORTES, 2007), when RBFs were considered, the diffusional and the classical forms of Black-Scholes equation led to the same results. Thus, this fact will not be shown in the figures to come.

In this work, the diffusional finite difference form of the BS equation is considered and, this approach was not considered previously. As shown in Figure 1, excellent solutions can be obtained by the PMGV technique (the results of the RBF method are not shown, because they do agree with the PMGV solution. Figure 1 was obtained with Nt =100 time steps, N =112 meshes and S = 0.714, with an upper value of S equal to 80. With these parameters, the cubic RBF led to an option price relative error of 0.00039% at the exercise option value (E=50), while in the case of the TPS RBF, the relative error was 0.019%. The diffusional finite difference method led to an error of 0.0346%. Thus, under controlled and optimal conditions, all methods led to very accurate solutions.

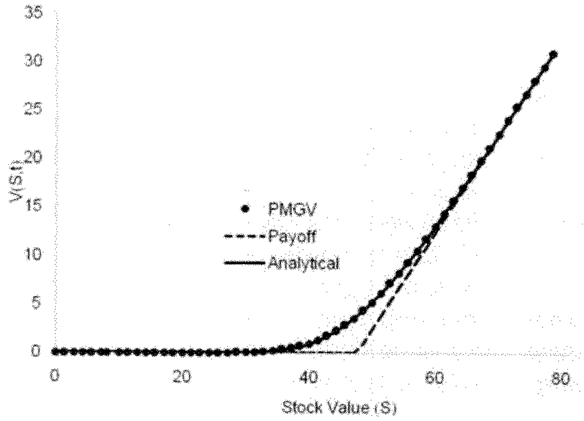

FIGURE 1. Cubic RBF and PMGV simulated values of a call option, V, compared against the analytical solution and payoff function values; E = 50, T = 1, = 20%, r = 5%.

Figure 2 shows the effect of the integration scheme; the ?-variation was performed based on Nt = 100, N = 112, Smax = 80 for cubic RBF and Nt = 300, N = 200 and Smáx = 100, in the case of TPS RBF. As can be noted, it is advisable to use implicit schemes with ? ? 0.5, as a general rule; smaller ? values lead to divergence. Although the choice of ? can affect the accuracy of numerical solutions, Figure 2 allows affirming that a good choice is to use a ? value slightly larger than the one that leads to divergence. Even higher values of ? still provide highly accurate solutions. One can notice that Cubic RBF is more efficient than TPS RBF, since cubic RBF require less time steps, a smaller Smax and less meshes for approximately similar error levels.

Changes in S, t and Smax, that is, the other simulation parameters, did not affect the just mentioned conclusions.

On the other hand, and very differently from Marques and Fortes' (2007) results, the PMGV scheme led to stable solutions, at the cost of much less accurate results (relative error slightly less than 1%). This radically differs from the very accurate errors obtained with the RBF methods. Thus, the PMGV scheme showed itself to be very stable and not to show a large divergence region as the other two methods, with respect to the integration method.

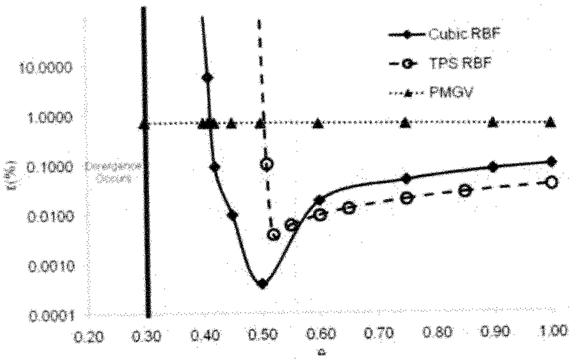

FIGURE 2. Relative errors of cubic and TPS RBF as affected by the integration ?-value. E = 50, T = 1, = 20%, r = 5%; Nt = 100, N = 112, Smax = 80 for cubic RBF and Nt = 300, N = 200 and Smax = 100, in the case of TPS RBF.

The effect of the number of time steps on the accuracy of the numerical solutions can be visualized in Figure 3. Again, the PMGV scheme shows a stable performance at the cost of very low accuracy, as compared to the RBF methods. As can be noticed, Cubic RBF performs better than TPS RBF. TPS error decreases as the time step (= T/Nt) decreases or the number of time steps increase; however, if the number of time steps gets lower than a limiting value (300), divergence occurs. On the other hand, cubic RBF lead to smaller errors and requires less times steps and presents a point of minimum error. Thus, as a general rule, TPS RBF should be used with larger number of time steps or smaller time steps, while the optimal number of time steps for cubic RBF can be obtained by starting with a large number of times steps and reducing it by increasing the time step and observing the solution behavior.

Figure 4 shows the numerical solution behavior of both RB functions, as affected by the number of grid points or, inversely, the stock value mesh size. As can be noticed, TPS RBF diverges when the number of grid points exceeds 200; interestingly enough, at this grid point number, it reaches the smallest relative error. Cubic RBF behave excellently well, with very small relative option value errors. Proper choice of Cubic RBF involves checking the solution consistency at a larger number of grid points. Thus, Figure 4 allows recommending the use of the maximum acceptable value of grid points, as a general rule since it leads to accurate values. In the investigated literature, there is no known technique for finding the minimum error point for the RBF method. And, again, the PMGV scheme led to very high error levels. Thus, if one is interested in accuracy, the PMGV scheme is not advisable, if used under conditions far from the optimum.

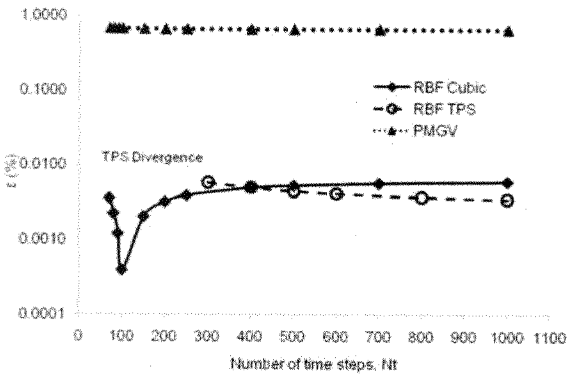

FIGURE 3. Relative errors of TPS and Cubic RBF call option values at the exercise option value (%) as affected by the number of time steps, Nt. E = 50, T = 1, = 20%, r = 5%; N = 112, S_{max} = 80 for cubic RBF and N = 200 and S_{max} = 100, in the case of TPS RBF.

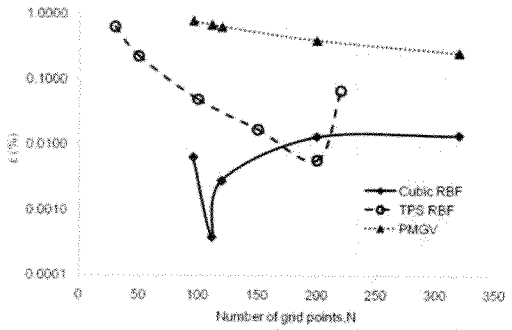

FIGURE 4. Relative errors of TPS and Cubic RBF call option values at the exercise option value (%) as affected by the number of grid points, N. E = 50, T = 1, = 20%, r = 5%; Nt = 100, S_{max} = 80 for cubic RBF and Nt = 300 and S_{max} = 100, in the case of TPS RBF.

Figure 5 shows that there is a value of S_{max} that minimizes the relative error of the numerical solutions through any of the RB functions; however low values can increase the error or lead to divergence. So, the numerical resFIGURE 5. Relative error at exercise option value (%) vs. Maximum Stock Value, Smax.

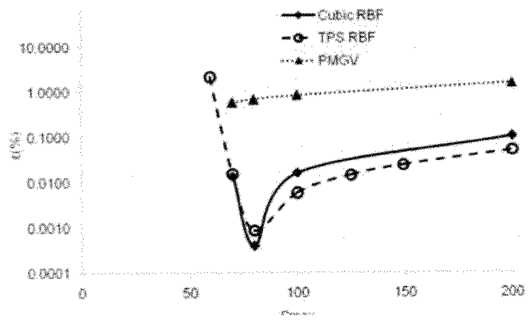

FIGURE 5. Relative error at exercise option value (%) vs. Maximum Stock Value, Smax.

Parametric sensitivity analysis

Figure 6 shows the effect of the volatility value on the accuracy of the numerical methods. One can notice that, if 1% is defined as an acceptable error level:

- The TPS RBF presents convergence and acceptable accuracy only for values up to 20%.
- The PMGV scheme presents acceptable accuracy in the 20-40% -range.
- The Cubic RBF presents the largest acceptable range of values: 1-50%.

These -parameter effects can also be visualized in Figure 7.

Figure 8 shows the effect of the risk-free interest rate on the accuracy of the numerical methods. One can notice that, if 1% is defined as an acceptable error level:

- The TPS RBF presents convergence and acceptable accuracy range of r values:1-30% ; from this point, errors tend to increase exponentially.
- The PMGV scheme presents excellent accuracy in the 1-10% r-range.
- The Cubic RBF and the PMGV scheme present the large acceptable range of r values: 1-40%.

FThese r-parameter effects can also be visualized in Figure 9.

Thus, it has been shown that each Black-Scholes solution should be optimized for the given expected values of volatility and risk-free interest rate, due to the sensitivity of the numerical methods. The simulations herein performed were based on optimal values discussed in Figure 1.

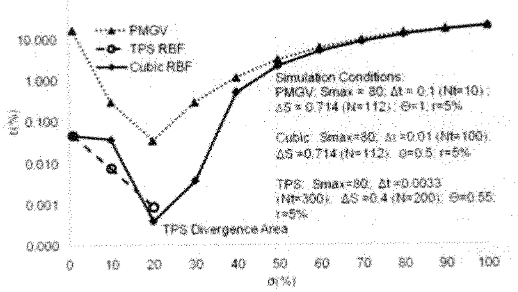

FIGURE 6. Effect of the volatility value on the accuracy of the numerical methods.

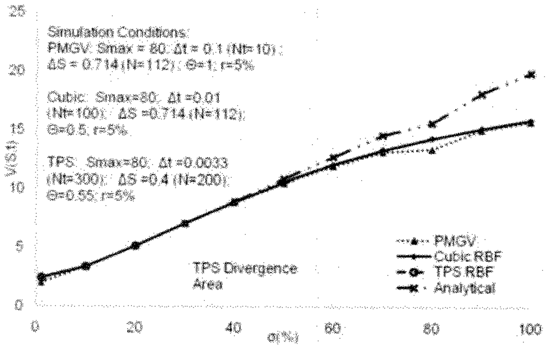

FIGURE 7. Effect of the volatility value on the numerically evaluated premium (option price).

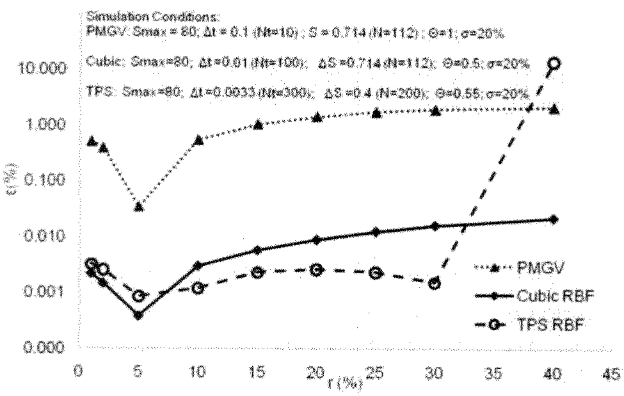

FIGURE 8. Effect of the risk-free interest rate on the accuracy of the numerical methods.

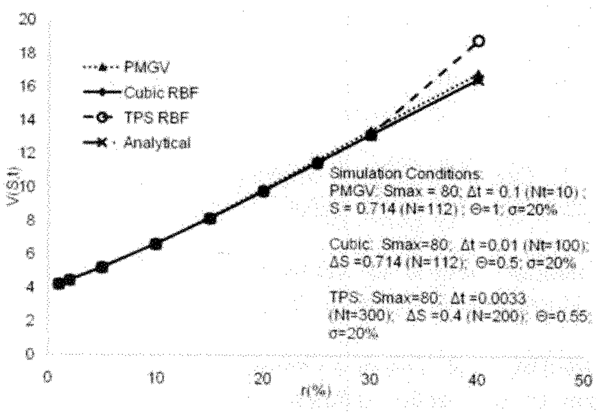

FIGURE 9. Effect of the risk-free interest rate on the numerically evaluated premium (option price).

Conclusions

A study of optimized numerical solutions of Black-Scholes equation is presented in detail. The work of Marques and Fortes (2007) was extended so as to include results of the diffusional finite difference method. The PMGV scheme is shown to lead to non-accurate results, if applied away from its optimum simulation parameters: S, t, Smax; it is also less sensitive to divergence and allows working in a broader range than the RBF methods. On the other hand, RBFs allow much more accurate solutions, but are subject to divergence. TPS RBF is shown to be particularly sensitive do divergence; when it does not diverge, the associated solutions are accurate.

Based on the parametric sensitivity analysis, the following conclusions can be stated:

1. TPS RBF should be well calibrated for given volatility and interest rate values; otherwise, small changes in these parameters may lead to divergence or wrong results.
2. Cubic radial functions and the PMGV scheme furnish accurate results to the BS equation, with a broad variation of r and values.

Thus, the numerical methods herein proposed are accurate and amenable to be used to solve Black and Scholes equation. Special care should be taken when variations of the interest rate or average volatility occur or tend to occur.

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References

BARUCCI, E.; LANDI, L.; CHERUBINI, U. Computational Methods in Finance: Option Pricing. IEEE Computational Science & Engineering, pp. 66-80. 1996.

BLACK, F.; SCHOLES, M. The pricing of options and corporate liabilities. Journal of Political Economy, v. 81, p. 637-654, 1973.

BOZTOSUN, I.; CHARAFI, A. An analysis of the linear advection-diffusion equation using mesh-free and mesh-dependent methods. Engineering Analysis with Boundary Elements, v. 26, pp. 889-895, 2002.

BROWM, D.; LING, L.; KANSA, E.; LEVESLEY, J. On approximate cardinal preconditioning methods for solving PDEs with radial basis functions. Engineering Analysis with Boundary Elements, v. 29, p. 343-353, 2005.

CONT, R.; VOLTCHKOVA, E. A finite Difference Scheme for option pricing in jump diffusion and exponential lévy models. Society for Industrial and Applied Mathematics, v. 43, n. 4, p. 1596-1626, 2005.

COX, J.; RUBINSTEIN, M. Options Markets. Englewood Cliffs: Prentice-Hall, 1985. 498 p.

CRETIEN, P. D. Comparing option pricing models. Trading Techniques, v.35 n.11, p. 38-41, sept. 2006

FORTES, M. The diffusional method for convection-diffusion equations: finite element one-dimensional solutions. Numerical Methods in Thermal Problems, Swansea, vol. 10, p. 57-68, 1997.

FORTES, M. and FERREIRA, W.R. The one dimensional transient diffusional method: finite element adaptive solutions to convection-diffusion problems. International Journal of Thermal Sciences, Paris, v. 38, p.780-796, 1999.

FORTES, M. and FERREIRA, W.R. Finite volume approach to transport equations by the diffusional method. In: SIMPÓSIO DE MECÂNICA COMPUTACIONAL, 3. 1998. Ouro Preto. Anais ...Vol. I, p. 443-452, 1998.

FORTES, M.; FERREIRA, W. R.; BARI, M. L.; YOSHITAKE, M. The Diffusional Method Applied to Financial Engineering. In: XXIX ENANPAD, Brasília, Anais do ENANPAD, CD-ROM. 2005.

GARCIA-OLIVARES, A. Analytical solution of nonlinear partial differential equations of physics. Emerald Research Register, v. 32, n. 4, p. 548-560, 2003.

KARAGEORGHIS, A.; CHENB C.S.; SMYRLIS Y-S. A matrix decomposition RBF algorithm: Approximation of functions and their derivatives. Applied Numerical Mathematics, pp. 1-16

KOC, M. B.; BOZTOSUN, I.; BOZTOSUN, D. On the Numerical Solution of Black-Scholes Equation. In: International Workshop on Mesh Free Methods, 2003.

HOFFMAN, J.D. Numerical Methods for Engineers and Scientists. New York: McGraw-Hill, 1992. 825p.

HULL, J. Options, Futures and other Derivatives Securities. Englewood Cliffs: Prentice-Hall, 1989. 341 p.

LEENTVAAR, C.C.W. Numerical solution of the Black-Scholes equation with a small number of grid points. Amsterdam: Delft University of Technology, 2003, 88 p. (MS Thesis).

MARQUES, V.M.P.; FORTES, M. The radial basis function method applied to financial engineering. Reuna - Revista de Economia, Administração e Turismo, v.11, n2, p.11-22, 2006.

MEYER, R.; JOHNSON, N. Is Black-Scholes Irrelevant? The Numerical Algorithms Group, Accessed on February, 13th, www.nag.com, 5p., 2002.

MEYER, G.H. The numerical valuation of options with underlying jumps. Acta Math. Univ. Comenianae, v. 67, n.1, p. 69-82, 1998.

MUCHMORE P.; ERICKSON B.; GONZALEZ A.; FERRI, E.; KIEFER S. Black Scholes Option Valuation with the Method of Lines, Final Report, Santa Barbara: USCB Mathematics Department, 2005, 5p.

MURPHY, D.; PRENTER, P.M. Higher order methods for convection-diffusion problems. Computers and Fluids, Vol. 13, p.157-176, 1985.

PATANKAR, S.V. Numerical Heat Transfer and Fluid Flow. New York: Hemisphere, 1980. 197p.

SARRA, S.A. Adaptive radial basis function methods for time dependent partial differential equations. Applied Numerical Mathematics, v.54, pp. 79-94, 2005.

SIEGEL, J.G.; SHIM, J.K.; HARTMAN, S.W. The McGraw-Hill Pocket Guide to Business Finance - 201 Decision-Making Tools for Managers. New York: McGraw-Hill,1992.

WILMOTT, P.; HOWISON, S.; DEWYNNE, J. The Mathematics of Financial Derivatives: A Student Introduction. Cambridge: Cambridge University Press, 1995. 317 p.

WILMOTT, P. Derivatives: The Theory and Practice of Financial Engineering. New York: John Wiley & Sons, 1998. 741 pp.

ZHANG, C. Pricing American options by adaptive finite element method. Annual Report, Mathematics Department, University of Maryland, College Park. 2005. 7p.

ZIENKIEWICZ, O.C.; TAYLOR, R.L. The Finite Element Method. Vol.2 - Solid and fluid mechanics: dynamics and non-linearity. 4th Ed. London: McGraw-Hill, 1991. 807 p.