ARQUIVO 1

Numerical Solution and Analysis of Some Path-Dependent Derivatives in Financial Engineering

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Resumo

Este trabalho apresenta uma técnica numérica baseada no método difusional para analisar opções de barreira através da equação de Black-Scholes. Os modelos mais importantes de engenharia financeira são baseados nas equações de Black-Scholes e são usados para predizer o retorno de opções financeiras e, assim, ajudar nos processos de tomada de decisão. As equações de Black-Scholes são equações parabólicas hiperbólicas, com variáveis e parâmetros estocásticos. Adaptações no modelo original levaram a um conjunto de equações diferenciais parciais não lineares essencialmente equivalentes à equação de convecção-difusão da engenharia. Este trabalho mostra a aplicação do método difusional de diferenças finitas, proposto recentemente, para resolver as equações fundamentais de Black-Scholes, quando aplicada a várias opções financeiras dependentes do caminho (tipo de aplicação no tempo). Por comparação com soluções analíticas disponíveis, os resultados mostram que o método difusional permite analisar de maneira acurada opções de barreira dependentes do caminho e outros problemas correlatos de engenharia financeira. Devido à sua simplicidade e acurácia, o método de diferenças finitas difusional compete favoravelmente com outros esquemas de diferenças finitas.

Palavras-chave: Engenharia financeira - Equações de Black-Scholes - Opções de barreira - Opções dependentes do caminho - Método difusional - Equações diferenciais

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Abstract:

This paper presents a numerical technique based on the diffusional method for analyzing financial barrier options by means of Black-Scholes equation. The most important models of financial engineering are based on Black-Scholes equations, and are used to predict the outcome of financial options and, thus, help in decision-making processes. Black-Scholes equations consist of a set of parabolic hyperbolic equations, with stochastic variables and parameters. Improvements on the original model lead to a set of non-linear partial differential equations essentially equivalent to the engineering convection-diffusion equation. This paper shows the application of the recently developed diffusional finite difference method to solve the fundamental Black-Scholes equations, as applied to several path-dependent financial options. By comparison with available analytical solutions, the results show that the diffusional method allows to accurately analyze path-dependent barrier options and other related problems of financial engineering. Due to its simplicity and accuracy, the diffusional finite difference method competes very favorably with other finite difference schemes.

Keywords: Financial engineering - Black-Scholes´ equations - Barrier options - Path-dependent options - Diffusional method - Differential equations

Introduction

Only simple contracts in stock markets can be handled in a semi-quantitative way. The most important models of financial engineering are based on Black-Scholes equations, and are used to predict the outcome of financial options and derivative securities and, thus, help in decision-making processes (Cox & Rubinstein, 1985). The Black-Scholes option-pricing model is used to determine the expected value of an option. It provides insight into the valuation of debt relative to equity (Hull, 1989; Siegel et al., 1992). Black-Scholes basic equation is a linear parabolic hyperbolic equation, with stochastic variables and parameters.

Several finite difference methods have been proposed to solve the cited model and similar convection-diffusion equations, with varying degrees of success (Hoffman, 1992; Murphy & Prenter, 1985; Neuman, 1984; Smith, 1978; Wilmott, 1998; Wilmott et al., 1995). Different weighted residual methodologies are also available (Lee et al., 1987; Yu & Heinrich, 1986; Zienkiewicz & Taylor; 1991).

The diffusional method was recently proposed (Fortes, 1997; Fortes & Ferreira, 1999; Fortes & Ferreira, 1998) and is based on transforming the original hyperbolic parabolic partial differential equation into a parabolic partial differential equation; further application of finite difference formulation leads to a tridiagonal form, readily amenable to computer implementation. The method is simple to apply and will be shown to perform much better when solving benchmark and practical problems, than the commonly employed finite difference techniques (implicit finite-difference methods including Crank-Nicolson, Douglas schemes, ADI and Hopscotch methods; see Hoffman (1992) for limitations on these methods).

In a recent paper the diffusional finite difference method was applied to analyze derivatives in financial engineering, with special attention to Black-Scholes call option equation (Fortes et al., 2000a-b). Up to present no time dependent jump condition associated to Black-Scholes equation has been analyzed in conjunction with the diffusional method. Thus, this paper aims at presenting the diffusional method as applied to the analysis of path dependent options with particular reference to Barrier Options.

2. Background

2.1 Some basic aspects and definitions of financial engineering

Calls and puts are the two simplest forms of financial options. A call option is the right to buy a particular asset for an agreed amount at a specified time in the future. As an example, consider the following call option on a stock of a industry. It gives the holder the right to buy one of the industry stock for an amount $90 in one month's time. Today's stock price is $85. The amount '90' which the purchaser can pay for the stock is called the exercise price or strike price. The date on which he must exercise his option, if he decides to, is called the expiry date. The stock on which the option is based is known as the underlying asset. He would exercise the option at expiry if the stock is above the strike and not if it is below. If S means the stock price and E the strike then at expiry, t = T, the option is worth

$$M(S) = \max(S - E, 0) \tag{1}$$

where M(S) is the option value at expiry, or, in other words, M(S) is a function of the underlying asset and is called the payoff function. At any other time t, the option value will be called V(S,t); in this way, V(S,T) = M(S).

A put option is the right to sell a particular asset for an agreed amount at a specified time in the future. The holder of a put option wants the stock price to fall so that he can sell the asset for more than it is worth. The payoff function for a put option is

$$V(S,T) = \max(E - S, 0) \tag{2}$$

The option is only exercised if the stock falls bellow the strike price.

One of the most interesting features of calls and puts is that they have a non-linear dependence on the underlying asset. This non-linearity is very important in the pricing of options.

Other terms used to describe contracts with some dependence on a more fundamental asset are derivatives or contingent claims. In this paper, only options based on call options will be considered since put options can be treated similarly.

Figure 1 shows the value of a call option as a function of the underlying, at expiry (payoff curve or diagram), t = T and at a time t < T.

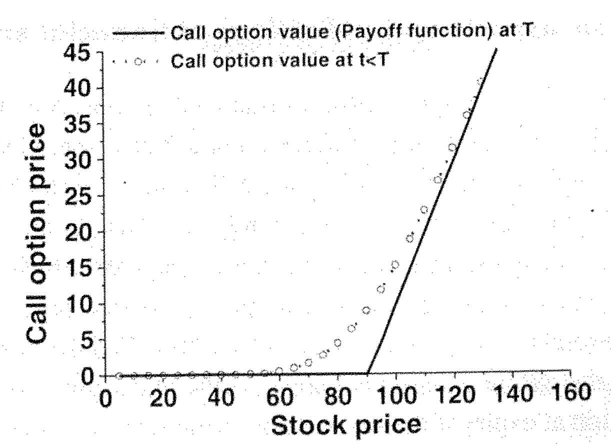

Figure 1 - Diagram for a call option.

A payoff diagram tells about what happens at expiry, how much money the option contract is worth at that time. The main objective of financial engineering, when options are at stake, is to define the function V(S, t), since this function will inform the value of the option at any time t, and thus, allows buying or selling an option at a desired acceptable value.

There are several models that lead to estimates of V(S, t); the most important models are based on Black-Scholes equations. The original and most famous Black-Scholes equation is the following stochastic-deterministic equation, whose main stochastic parameter is the volatility (akin to standard deviation) of the underlying asset, s, and the practiced interest rate, r:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{3}$$

The Black-Scholes equation is a linear parabolic partial differential equation and is similar to the convection-diffusion equation found in fluid mechanics.

2.2 Barrier Options

By choosing the right options, such as barrier options, among others, a purchaser can reduce his risks. Barrier options have a payoff that is contingent on the underlying asset reaching some specified level before expiry. There are two main types of barrier option (Wilmott, 1998):

- The out option, that only pays off if a level is not reached. If the barrier is reached then the option is said to have knocked out.

- The in option, that pays off as long as a level is reached before expiry. If the barrier is reached then the option is said to have knocked in.

A barrier option can also be characterized by the position of the barrier relative to the initial value of the underlying:

- If the barrier is above the initial asset value, one has an up option.

- If the barrier is below the initial asset value, one has a down option.

The main boundary conditions for the most common barrier options are discussed now.

a) Up-and-out barrier option with the barrier level at $S = S_u$: In this case, the Black-Scholes equation must be solved for $0 \le S \le S_u$.

$$V(S_u, t) = 0 \quad \text{for} \quad t < T \tag{4}$$

and, if the barrier is not triggered

$$V(S, T) = \max(S - E, 0) \tag{5}$$

b) Up-and-in barrier option with the barrier level at $S = S_u$: If the option is an up-and-in contract then on the upper barrier the contract must have the same value as a call option contract. In this case, the Black-Scholes equation must be solved for $S_u \le S \le \infty$.

$$V(S_u, t) = \max(S - E, 0) \qquad S_u \le S \le \infty \tag{6}$$

$$V(S, T) = 0 \qquad 0 \le S < S_u \tag{7}$$

c) Down-and-out barrier option with barrier level at $S = S_d$. In this case, for $S_d \le S < \infty$, if the barrier is not triggered

$$V(S, T) = \max(S - E, 0) \tag{8}$$

and, if the barrier is triggered.

$$V(S_d, t) = 0 \tag{9}$$

d) Down-and-in barrier option with the barrier level at $S = S_d$. If the option is a down-and-in contract, then on the lower barrier the contract must have the same value as a call option contract. Thus

$$V(S_d, t) = \max(S - E, 0) \qquad \text{for } 0 \le S \le S_d \tag{10}$$

$$V(S, t) = 0 \qquad \text{for } S_d \le S < \infty \tag{11}$$

It is important to remind that for a simple call option the boundary conditions are for $0 \le S < \infty$.

3. Methodology

3.1 The diffusional method applied to Black-Scholes equation

In this work, the Diffusional Method was applied to solve equation (3). This equation must be solved with a final condition depending on the payoff: each contract will have a different functional form prescribed at expiry $t = T$, depending on whether it is a call, a put or something more fancy. The final condition must be imposed to make the solution unique. In order to make it an initial boundary value problem let us make $\tau = T - t$, so that equation (3) becomes

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0 \tag{12}$$

In order to put this last equation in the form expressed by a convection-diffusion equation and amenable to the diffusional method, use is made of the identity

$$\frac{\partial}{\partial S}\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S}\right) = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \sigma^2 S \frac{\partial V}{\partial S} \tag{13}$$

in equation (12), so that one obtains

$$\frac{\partial V}{\partial \tau} - \frac{\partial}{\partial S}\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S}\right) + \left(\sigma^2 - r\right)S\frac{\partial V}{\partial S} + rV = 0 \tag{14}$$

Equation (14) is then similar to the transient one-dimensional convection-diffusion equation:

$$\frac{\partial V}{\partial \tau} + u\frac{\partial V}{\partial S} - \frac{\partial}{\partial S}\left(\Gamma\frac{\partial V}{\partial S}\right) + Q = 0 \tag{15}$$

where one can easily identify

$$u = \left(\sigma^2 - r\right)S \;; \quad \Gamma = \frac{1}{2}\sigma^2 S^2 \;; \quad Q = rV \tag{16}$$

3.2 The diffusional finite difference method for solving Black-Scholes equation

By assuming to be constant or an average within the integration range, Fortes (1997) and Fortes and Ferreira (1999) showed that the above equation (15) can be put into the following diffusional form

$$e^{-\frac{2\text{Pe}\,S}{\Delta S}}\frac{\partial V}{\partial \tau} - \frac{\partial}{\partial S}\left(\Gamma e^{-\frac{2\text{Pe}S}{\Delta S}}\frac{\partial V}{\partial S}\right) + e^{-\frac{2\text{Pe}\,S}{\Delta S}}Q = 0 \qquad (17)$$

where Pe is a Financial Peclet number, $\text{Pe} = \frac{u\Delta S}{2\Gamma}$, or $\text{Pe} = \frac{(\sigma^2 - r)\Delta S}{\sigma^2 S}$ and DS is the total domain of the problem, or the mesh size, in the case of numerical schemes.

The above equation (17) is in an excellent form to be solved by any numerical technique, and more particularly, by the finite difference method.

Consider the one-dimensional control volume shown in Figure 2. Finite difference approximations to the exact partial derivatives appearing in the partial differential equation must be developed. For simplicity of notation Si-Si-1 = Si+1-Si=DS is denoted by h, where is now clearly depicted as a characteristic (spot market price) length (in this case, the mesh size).

Figure 2 - One-dimensional control volume

Using a central difference approximation and assuming uniform grid spacing the diffusive term of equation (17) can be written as

$$\frac{\partial}{\partial S}\left(\Gamma e^{-\frac{2\text{Pe}S}{h}}\frac{\partial V}{\partial S}\right)_i \approx \frac{1}{h}\left[\left(\Gamma e^{-\frac{2\text{Pe}S}{h}}\frac{\partial V}{\partial S}\right)_{i+1/2} - \left(\Gamma e^{-\frac{2\text{Pe}S}{h}}\frac{\partial V}{\partial S}\right)_{i-1/2}\right] \qquad (18)$$

where the subscript i + ½ indicates an evaluation at the point midway between Si and Si+1 and the subscript i - ½ is defined similarly. Thus one can write

$$\frac{\partial}{\partial S}\left(\Gamma e^{-\frac{2\text{Pe}S}{h}}\frac{\partial V}{\partial S}\right) \approx \frac{1}{h}\left[\left(\Gamma e^{-\text{Pe}}\frac{\partial V}{\partial S}\right)_{i+1/2} - \left(\Gamma e^{\text{Pe}}\frac{\partial V}{\partial S}\right)_{i-1/2}\right] \qquad (19)$$

Again, by assuming constant local properties, replacing the appropriate derivatives by central difference approximations, and rearranging, one obtains

$$\frac{\partial}{\partial S}\left(\Gamma e^{-\frac{2\text{Pe}S}{h}}\frac{\partial V}{\partial S}\right) \approx \frac{\Gamma}{h^2}\left[e^{-\text{Pe}}V_{i+1} - \left(e^{-\text{Pe}} + e^{\text{Pe}}\right)V_i + e^{\text{Pe}}V_{i-1}\right] \qquad (20)$$

Applying the log-mean difference concept to the exponential term multiplying Q, and assuming Q constant or an average within the element, the source term can be written as

$$e^{-\frac{2\text{Pe}\,S}{h}}Q \approx \frac{e^{\text{Pe}} - e^{-\text{Pe}}}{2\text{Pe}}Q \qquad (21)$$

The transient term is approximated by a one-sided difference for the first derivative with the log-mean difference concept being applied to the exponential term,

$$e^{-\frac{2\text{Pe}S}{h}}\frac{\partial V}{\partial \tau} \approx \left(\frac{e^{\text{Pe}} - e^{-\text{Pe}}}{2\text{Pe}}\right)\left(\frac{V_i^{n+1} - V_i^n}{\Delta \tau}\right) \qquad (22)$$

where n stands for time plane and Dt is the time interval. Making use of the mean value theorem, equations (20, 21 and 22) can be substituted into equation (15), resulting in a set of algebraic equations whose ith-equation is given by

$$\begin{aligned}&\left(\frac{V_i^{n+1} - V_i^n}{\Delta\tau}\right) + \theta\left(-\frac{2\text{Pe}\Gamma}{h^2}\left[\left(\frac{e^{\text{Pe}}}{e^{\text{Pe}} - e^{-\text{Pe}}}\right)V_{i-1}^{n+1} - (\coth \text{Pe})V_i^{n+1} + \left(\frac{e^{-\text{Pe}}}{e^{\text{Pe}} - e^{-\text{Pe}}}\right)V_{i+1}^{n+1}\right] + Q^{n+1}\right)\\ &+ (1-\theta)\left(-\frac{2\text{Pe}\Gamma}{h^2}\left[\left(\frac{e^{\text{Pe}}}{e^{\text{Pe}} - e^{-\text{Pe}}}\right)V_{i-1}^{n} - (\coth \text{Pe})V_i^{n} + \left(\frac{e^{-\text{Pe}}}{e^{\text{Pe}} - e^{-\text{Pe}}}\right)V_{i+1}^{n}\right] + Q^{n}\right) = 0\end{aligned} \qquad (23)$$

where $0 \le \theta \le 1$. The parameter q is a weighted average of the explicit ($\theta = 0$) and implicit ($\theta = 1$) methods. The implicit and explicit schemes have a truncation error $O(d\tau)$ and the Crank-Nicolson scheme, $\theta=1/2$, an error $O(d\tau^2)$. Because of this, in financial engineering, Crank-Nicolson scheme is more used. Recently, researchers have used the Douglas scheme, $\theta=\frac{1}{2}-\frac{dS^2}{12d\tau}$, that leads to an improvement in the order of accuracy (Wilmott, 1998).

In order to facilitate comparison with other numerical methods, a local parameter α, defined by $\alpha = \coth Pe - 1/Pe$, is substituted into equation (23), leading after algebraic manipulations to

$$\left(\frac{V_i^{n+1}-V_i^n}{\Delta\tau}\right)+\theta\left(-\frac{\Gamma}{h^2}[Pe(\alpha+1)+1]V_{i-1}^{n+1}+\frac{2\Gamma}{h^2}[Pe\alpha+1]V_i^{n+1}-\frac{\Gamma}{h^2}[Pe(\alpha-1)+1]V_{i+1}^{n+1}+Q^{n+1}\right)$$
$$+(1-\theta)\left(-\frac{\Gamma}{h^2}[Pe(\alpha+1)+1]V_{i-1}^n+\frac{2\Gamma}{h^2}[Pe\alpha+1]V_i^n-\frac{\Gamma}{h^2}[Pe(\alpha-1)+1]V_{i+1}^n+Q^n\right)=0 \tag{24}$$

Regardless of which value is specified for q, equation (24) can be rewritten in the form:

$$\begin{aligned} a_s\theta V_{i-1}^{n+1}+(b_t+b_s\theta)V_i^{n+1}+c_s\theta V_{i+1}^{n+1} &= \\ -d_i-a_s(1-\theta)V_{i-1}^n+(b_t-b_s(1-\theta))V_i^n-c_s(1-\theta)V_{i+1}^n & \end{aligned} \tag{25}$$

with the following coefficients,

$$a_s=-Pe(\alpha+1)-1;\quad b_s=2+2Pe\alpha\ ;\quad c_s=-Pe(\alpha-1)-1\ ; \tag{26}$$

$$b_t=\frac{2Pe}{C}\ ,\quad d_i=\frac{h^2}{\Gamma}\left[\theta Q^{n+1}+(1-\theta)Q^n\right] \tag{27}$$

where Courant number, C, is defined as $C=\frac{u\Delta\tau}{h}$.

Equation (25) represents a set of equations that can be solved to give the options values for all possible S values and time values.

4. Results and Discussion

Figures below show the numerical solutions of Black-Scholes equation for call option pricing, and, down-and-out, down-and-in and up-and-out call options. The associated parameters were: E = 100, r = 10% and σ = 25%. The expiring time was taken to be T = 1, and the solution was obtained for both the call option value and the payoff function at t = 0. The solution was obtained by means of the fully explicit procedure ($\theta = 0$), and, thus, all V values at time t-Δt, could be obtained directly from the previous values at time t, without any recourse or need to solving a system of equations. For a simple **call** option the analytical solution is given by

$$V(S,T-t)=S\cdot N(d_1)-E\cdot N(d_2)\cdot e^{-r(T-t)} \tag{28}$$

where

$$d_1(S,T-t)=\frac{\ln\left(\frac{S}{E}\right)+\left(r+\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}} \tag{29}$$

$$d_2(S,T-t)=d_1(S,T-t)-\sigma\cdot\sqrt{T-t} \tag{30}$$

and N is the cumulative normal probability density function

$$N(x)=(2\pi)^{-\frac{1}{2}}\int_0^x e^{-\frac{1}{2}u^2}du+\frac{1}{2} \tag{31}$$

The instantaneous Payoff function, M, is given by

$$M(S,T-t)== \begin{cases} S-Ee^{-r(T-t)}, \text{ if } S-Ee^{-r(T-t)}\ge 0 \\ 0, \text{otherwise} \end{cases} \tag{32}$$

4.1 Down-and-out call option

Consider the down-and-out call option with barrier level S_d below the strike price E. The function $V_v(S,t)$ is the Black-Scholes value of a simple call option with the same maturity and payoff as the barrier option. The analytical solution is then given by (Wilmott, 1998)

$$V(S,t) = V_v(S_u,t) - \left(\frac{S}{S_d}\right)^{1-2r/\sigma^2} V_v\left(\frac{S_d^2}{S},t\right) \qquad (33)$$

The value of this option is shown as a function of S in Figure 3. The barrier level $S_d = 80$ and $\Delta S=1$ were used. Also shown is the value of simple call option. The numerical results do not differ significantly from the exact solution and thus, exact and numerical value curves overlap each other in the figure.

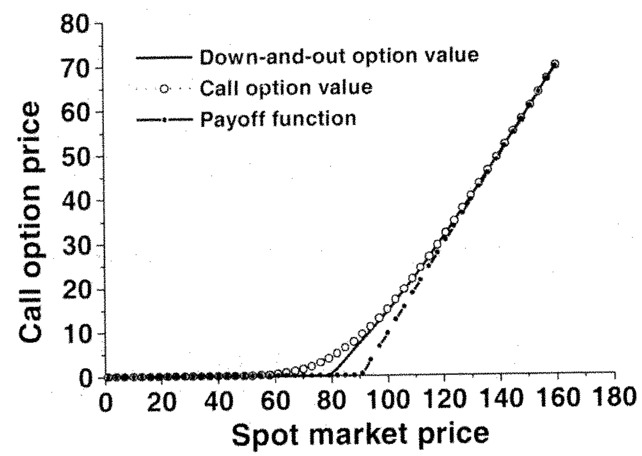

Figure 3 - Value of a down-and-out call option

4.2 Down-and-in Call Option

In the absence of any rebates the relationship between an 'in' barrier option and an 'out' barrier option (with same payoff and same barrier level) is

$$\text{in} + \text{out} = \text{simple call option} \qquad (34)$$

If the 'in' barrier is triggered then so is the 'out' barrier, so whether or not the barrier is triggered the simple call option payoff is still obtained at expiry. Thus, the value of a down-and-in cal option is

$$V(S,t) = \left(\frac{S}{S_d}\right)^{1-2r/\sigma^2} V_v\left(\frac{S_d^2}{S},t\right) \qquad (35)$$

The value of this option is shown as a function of S in Figure 4. The barrier level $S_d = 90$ and $\Delta S=5$ were used. Note that the values of down-and-in option and simple call option coincide at the barrier. Again, the numerical results do not differ significantly from the exact solution.

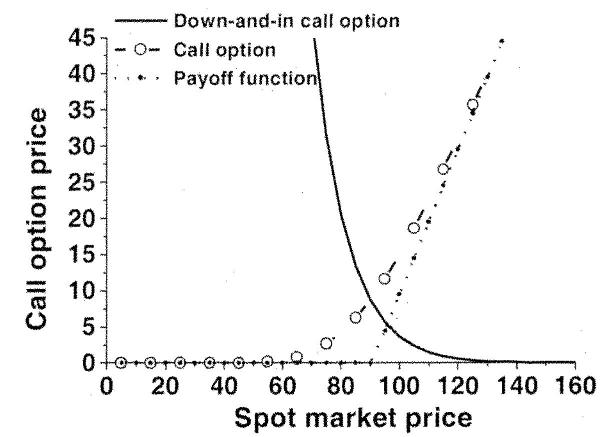

Figure 4 - Value of a down-and-in call option

4.3 Up-and-out Call Option

The barrier Su for an up-and-out call option must be above the strike price E (otherwise the option would be worthless). This makes the solution for the price more complicated.

The exact value of an up-and-out call option is (Wilmott, 1998)

$$\begin{aligned} V(S,T-t) = {} & S\big(N(d_1) - N(d_3) - b\big(N(d_6) - N(d_8)\big)\big) \\ & - Ee^{-r(T-t)}\big(N(d_2) - N(d_4) - a\big(N(d_5) - N(d_7)\big)\big) \end{aligned} \qquad (36)$$

where d_1 and d_2 are given by equations (29-30) and

$$a=\left(\frac{S_b}{S}\right)^{-1+\frac{2r}{\sigma^2}} \quad \text{and} \quad b=\left(\frac{S_b}{S}\right)^{1+\frac{2r}{\sigma^2}} \tag{37}$$

$$d_3(S,T-t)=\frac{\ln\left(\frac{S}{S_b}\right)+\left(r-\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}}$$

$$d_4(S,T-t)=\frac{\ln\left(\frac{S}{S_b}\right)+\left(r+\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}}$$

$$d_5(S,T-t)=\frac{\ln\left(\frac{S}{S_b}\right)-\left(r+\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}}$$

$$d_6(S,T-t)=\frac{\ln\left(\frac{S}{S_b}\right)-\left(r-\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}} \tag{38}$$

$$d_7(S,T-t)=\frac{\ln\left(\frac{SE}{S_b^2}\right)-\left(r-\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}}$$

$$d_8(S,T-t)=\frac{\ln\left(\frac{SE}{S_b^2}\right)-\left(r+\frac{1}{2}\sigma^2\right)\cdot(T-t)}{\sigma\cdot\sqrt{T-t}}$$

The value of this option is shown in Figure 5. The barrier level $S_u = 110$ and $\Delta S=5$ were used. Also shown are the values of simple call option and down-and-in call option, for comparison. Again, the numerical results do not differ significantly from the exact solution.

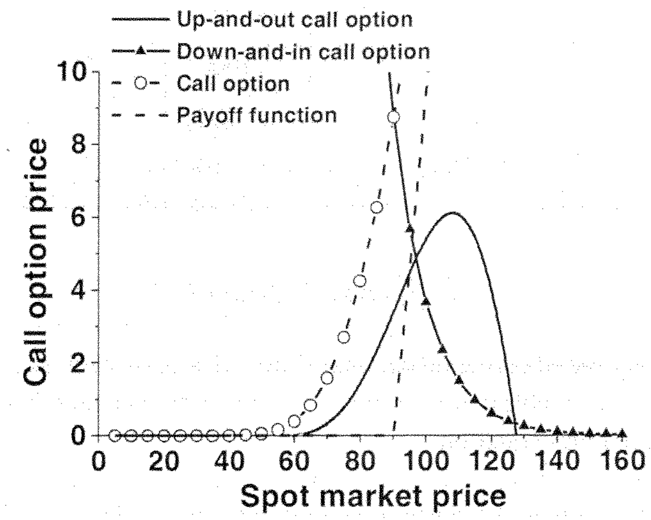

Figure 5 - Value of a up-and-out call option

4.4 Performance of the diffusional method

For the studied financial conditions, Pe and C are very small. Thus, Fourier number, $F=\frac{\Gamma\Delta\tau}{h^2}$, was the predominant dimensionless parameter affecting the numerical performance and accuracy. The numerical results agreed with the convergence criteria (Fortes & Ferreira, 1999) which states that convergence could be achieved whenever $F > 1/4$. It should be emphasized that whenever convergence was attained, the solutions were sound. It should also be noted that even coarse mesh refinements and rather large time-step led to rather acceptable solutions, as shown in Table I, in the case of simple call option. Similar results were obtained for all barrier option simulations.

Table I. Effect of temporal and mesh sizes on the numerical call option values at t =0

Δt	ΔS	Option value at S = 40 (Exact value = 5.9903)	Number of time steps	Number of meshes	Error (%)
10-4	0.5	6.0269	10000	160	0.61
10-3	1.0	6.0621	1000	80	1.2
10-2	2.0	6.1317	100	40	2.4
10-1	4.0	6.2943	10	20	5.1

5. Conclusions

A methodology based on the diffusional finite difference method is shown to be very effective and reliable to solve financial engineering problems that are governed by the Black-Scholes equation.

A change in the format of the Black-Scholes equation had to be realized in order to adapt it to the standard form of the diffusional method. In this way, other more complex modified forms of Black-Scholes equation become amenable to solutions by the diffusional method. Accurate solutions could be obtained by means of the fully explicit scheme, which does not require solving a usual simultaneous set of non-linear equations.

Numerical evaluation of three barrier options led to excellent, accurate results.

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